INEQUIVALENT REPRESENTATIONS OF COMMUTATOR OR ANTICOMMUTATOR RINGS OF FIELD OPERATORS AND THEIR APPLICATIONS

M. MATEJKA

DEPARTMENT OF CHEMICAL PHYSICS, FACULTY OF MATHEMATICS, PHYSICS AND COMPUTER SCIENCES, COMENIUS UNIVERSITY, MLYNSKÁ DOLINA, 842 48 BRATISLAVA, SLOVAKIA

M. NOGA

DEPARTMENT OF THEORETICAL PHYSICS, FACULTY OF MATHEMATICS, PHYSICS AND COMPUTER SCIENCES, COMENIUS UNIVERSITY, MLYNSKÁ DOLINA, 842 48 BRATISLAVA, SLOVAKIA

Abstract. Hamiltonian of a system in quantum field theory can give rise to infinitely many partition functions which correspond to infinitely many inequivalent representations of the canonical commutator or anticommutator rings of field operators. This implies that the system can theoretically exist in infinitely many Gibbs states. The system resides in the Gibbs state which corresponds to its minimal Helmholtz free energy at a given range of the thermodynamic variables. Individual inequivalent representations are associated with different thermodynamic phases of the system. The BCS Hamiltonian of superconductivity is chosen to be an explicit example for the demonstration of the important role of inequivalent representations in practical applications. Its analysis from the inequivalent representations' point of view has led to a recognition of a novel type of the superconducting phase transition.

PACS numbers: 03.70.+k, 05.30.-d, 11.10.-z, 74.20.Fg, 74.25.Bt, 74.78.Bz

1. Introduction

In quantum field theories based on operator formalism, the creation and annihilation field operators $a^\dagger_{\vec{k},\sigma}$ and $a_{\vec{k},\sigma}$ are the fundamental objects creating and annihilating resp. particles in quantum states denoted by quantum numbers $(\vec{k},\sigma)$, as for example, by the momentum $\vec{k}$ and the spin projection σ. In any quantum field theory the number of the operator $(a_{\vec{k},\sigma}, a^\dagger_{\vec{k},\sigma})$ pairs is infinite. These operators satisfy the canonical commutation or anticommutation relations

$$\left\{a_{\vec{k},\sigma}, a^\dagger_{\vec{k}',\sigma'}\right\} = \delta_{\vec{k},\vec{k}'}\delta_{\sigma,\sigma'} \ , \ \left\{a_{\vec{k}',\sigma'}, a_{\vec{k},\sigma}\right\} = \left\{a^\dagger_{\vec{k}',\sigma'}, a^\dagger_{\vec{k},\sigma}\right\} = 0 \ . \tag{1.1}$$

The operators $a_{\vec{k},\sigma}, a^\dagger_{\vec{k},\sigma}$ act on state vectors ψ which span a Hilbert space $\mathcal{H}$. In order to achieve a unique specification of the commutator or anticommutator ring of the operators (1.1), in addition to (1.1) one requires the existence of a vacuum state ϕ_0 in the Hilbert space $\mathcal{H}$ for which

$$a_{\vec{k},\sigma}\phi_0 = 0 \tag{1.2}$$

for all $(\vec{k},\sigma)$. In this case, the Hilbert space $\mathcal{H}$ is a space for a representation of the commutator or anticommutator ring (1.1) with the auxiliary condition (1.2).

As long as the number of the operators $a_{\vec{k},\sigma}, a^\dagger_{\vec{k},\sigma}$ entering the algebraic structure (1.1) and (1.2) is finite, there exists only one inequivalent representation for the algebraic relations (1.1) and (1.2). However, in quantum field theories describing systems with an infinite number of degrees of freedom, the algebraic structure (1.1) has infinitely many inequivalent representations [1]. Intuitively speaking, one can say that there exist infinitely many different and inequivalent

matrix realizations of the operators $a_{\vec{k},\sigma}$ and $a^{\dagger}_{\vec{k},\sigma}$ satisfying the same algebraic structure as (1.1) and (1.2). The situation reminds us very distantly of a Lie algebra of a non-compact group which has infinitely many unitary irreducible representations for its generators realized in forms of infinitely dimensional matrices. In contrast to the aforementioned Lie algebra with a finite number of its generators, the canonical ring (1.1) involves infinite number of the elements $a_{\vec{k},\sigma}$ and $a^{\dagger}_{\vec{k},\sigma}$ which can be realized by infinitely many different and inequivalent representations in forms of infinitely dimensional matrices.

The operators $a_{\vec{k},\sigma}$ and $a^{\dagger}_{\vec{k},\sigma}$ entering the ring (1.1) are assumed to form a complete set of operators which means that every operator in a quantum field theory can be built up out of them. Thus a grand canonical Hamiltonian H governing the dynamics of a given physical system is expressed as a given function of $a_{\vec{k},\sigma}$ and $a^{\dagger}_{\vec{k},\sigma}$, i.e.,

$$H = H(a^{\dagger}, a) \ . \tag{1.3}$$

The grand canonical partition function Z is expressed as the density matrix trace

$$Z = Tre^{-\beta H} \tag{1.4}$$

where $\beta = 1/T$ is the inverse temperature. All thermodynamic properties of the system are determined by the grand canonical potential

$$\Omega = -T \ln Z \ . \tag{1.5}$$

The statistical average values corresponding to physical observables associated with operators $A(a^{\dagger}, a)$ are determined by the relations

$$\langle A \rangle = \frac{1}{Z} Tr\{A(a^{\dagger}, a)e^{-\beta H}\} \ . \tag{1.6}$$

The grand canonical potential (1.5) and statistical average values (1.6) specify the so-called Gibbs state of the system governed by the given Hamiltonian (1.3).

Since the commutator or anticommutator ring (1.1) admits infinitely many inequivalent representations for the operators $a_{\vec{k},\sigma}$ and $a^{\dagger}_{\vec{k},\sigma}$, it implies that one has to associate the corresponding inequivalent representations to the Hamiltonian H, partition function Z and grand canonical potential Ω. Again, intuitively speaking, one can say that the matrix form of the Hamiltonian (1.3) is distinct for each inequivalent representation of the ring (1.1). This implies that the partition function (1.4), grand canonical potential (1.5) and statistical average values of physical observables (1.6) are distinct for each inequivalent representation of (1.1). In other words, the same Hamiltonian H gives rise to different results for Z, Ω and $\langle A \rangle$ corresponding to chosen inequivalent representations of the canonical ring (1.1). It implies that in each quantum field theory a given Hamiltonian can give rise to infinitely many Gibbs states. This theoretical conclusion seems to be in a conflict with the experience because every physical system resides always in a single Gibbs state for given values of thermodynamic variables, like temperature T, volume V and particle number N . The single Gibbs state corresponds to a single inequivalent representation of the commutator or anticommutator ring (1.1).

The answer how one should select an appropriate single inequivalent representation for a system at given values T, V and N out of infinitively many representations is uniquely given by the second law of thermodynamics. The second law of thermodynamics requires, e.g., the Helmholtz free energy $F(T, V, N)$ at given values of the thermodynamic variables T, V and N to be minimal with respect to any free parameters entering $F(T, V, N)$. Theoretically, it means that one should evaluate the Helmholtz free energies $F(T, V, N)$ for all inequivalent representations of (1.1) and select that single one which corresponds to their infimum at the given range of the thermodynamic variables T, V and N. A concrete example will demonstrate how it is done in practice.

For detailed understanding of inequivalent representations of the commutator or anticommutator ring (1.1) of field operators' role in practical applications we will study them from three different aspects. First, we will explicitly construct a certain class of inequivalent representations

of the anticommutator ring (1.1) of field operators. Second, we will show how one, in practical calculations, tacitly selects a single inequivalent representation by choosing an appropriate perturbation theory. At third, theoretical implications of inequivalent representations of electron field operator' anticommutator ring will be explicitly demonstrated on the BCS model Hamiltonian of superconductivity [2]. More specifically, a new class of inequivalent representations of (1.1), which has not been known till now and leads to a new unexpected superconducting state, will be constructed.

2. Inequivalent representations

For the purpose of practical applications to the BCS model Hamiltonian [2] and for the sake of simplicity, we consider a complete set of annihilation and creation operators $a_{k,\sigma}$ and $a_{k,\sigma}^\dagger$ of the fermion type. Let the index k run over integer numbers over the interval $k \in \left\langle -\frac{N}{2}, \frac{N}{2} \right\rangle$ and σ denotes spin 1/2 projection of a fermion, i.e., $\sigma = \downarrow, \uparrow = +, -$. In order to have a quantum field theory, we take the limit $N \to \infty$. The field operators $a_{k,\sigma}$ and $a_{k,\sigma}^\dagger$ satisfy the anticommutator ring

$$\left\{a_{k,\sigma}, a_{k',\sigma'}^\dagger\right\} = \delta_{k,k'}\delta_{\sigma,\sigma'} \ , \ \left\{a_{k',\sigma'}, a_{k,\sigma}\right\} = \left\{a_{k',\sigma'}^\dagger, a_{k,\sigma}^\dagger\right\} = 0 \ . \tag{2.1}$$

with the subsidiary condition

$$a_{k,\sigma}\phi_0 = 0 \tag{2.2}$$

on the vacuum state ϕ_0 for all (k, σ). The representation space for the anticommutator ring (2.1) with the subsidiary condition (2.2) can be chosen to be the Hilbert space $\mathcal{H}$ spanned by the basis vectors $\psi_{\{n_{k,\sigma}\}}$ defined by the formula

$$\psi_{\{n_{k,\sigma}\}} = \lim_{N\to\infty} \prod_{k,\sigma} \left(a_{k,\sigma}^\dagger\right)^{n_{k,\sigma}} \phi_0 \ , \tag{2.3}$$

where $n_{k,\sigma} = 0, 1$ are the occupation numbers of fermions in states (k, σ), and $\{n_{k,\sigma}\}$ denotes an infinite number array of items 0 and 1. Each such infinite array specifies one of the basis vectors of the Hilbert space $\mathcal{H}$.

Next we construct a class of inequivalent representations of the anticommutator ring (2.1) by adopting the same approach as outlined in Haag's work [1], however, for boson operators. We start from the operators $a_{k,\sigma}$ and $a_{k,\sigma}^\dagger$ obeying (2.1) and introduce the "unitary" transformations

$$\begin{aligned} c_{k,\sigma} &= e^{iQ} a_{k,\sigma} e^{-iQ} \ , \\ c_{k,\sigma}^\dagger &= e^{iQ} a_{k,\sigma}^\dagger e^{-iQ} \ , \end{aligned} \tag{2.4}$$

where Q is the Hermitian "operator"

$$Q = \lim_{N\to\infty} \sum_{k=-N/2}^{N/2} \alpha_k T_k \quad , \quad T_k = i\left(a_{k,+}^\dagger a_{-k,-}^\dagger - a_{-k,-} a_{k,+}\right) \tag{2.5}$$

and α_k are arbitrary real parameters. The anticommutation relations for the transformed operators $c_{k,\sigma}$ and $c_{k,\sigma}^\dagger$ are, of course, the same as given by (2.1). The operator e^{iQ} can be expressed as the following infinite product

$$e^{iQ} = \lim_{N\to\infty} \prod_{k=-N/2}^{N/2} \left[1 + iT_k \sin\alpha_k + T_k^2(\cos\alpha_k - 1)\right] \tag{2.6}$$

where

$$T_k^2 = 2a_{k,+}^\dagger a_{k,+} a_{-k,-}^\dagger a_{-k,-} - a_{k,+}^\dagger a_{k,+} - a_{-k,-}^\dagger a_{-k,-} + 1 \ . \tag{2.7}$$

The transformations (2.4), if evaluated through (2.4), are similar to the well-known Bogoliubov-Valatin transformations [5]

$$\begin{aligned} c_{k,+} &= u_k a_{k,+} + v_k a_{-k,-}^\dagger \quad , \quad c_{k,+}^\dagger = u_k a_{k,+}^\dagger + v_k a_{-k,-} \\ c_{k,-} &= u_k a_{k,-} - v_k a_{-k,+}^\dagger \quad , \quad c_{k,-}^\dagger = u_k a_{k,-}^\dagger - v_k a_{-k,+} \ , \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} u_k &= \cos\alpha_k \\ v_k &= \sin\alpha_k \ . \end{aligned} \tag{2.9}$$

In the limit $N \to \infty$, the operator e^{iQ} given by (2.4) is not a proper operator but transforms every vector ψ of the Hilbert space $\mathcal{H}$ into one $\psi^{'} = e^{iQ}\psi$ of a Hilbert space $\mathcal{H}^{'}$ with unexpected properties explained as follows. Let us denote by $\varphi_{\{n_k\}}$ any basis vector of $\mathcal{H}$ given by the formula

$$\varphi_{\{n_k\}} = \lim_{N\to\infty} \prod_{k=-N/2}^{N/2} \left(a^{\dagger}_{k,+} a^{\dagger}_{-k,-}\right)^{n_k} \phi_0 \tag{2.10}$$

where $n_k = n_{k,+} = n_{-k,-} = 0, 1$. All the basis vectors $\varphi_{\{n_k\}}$ form a subspace of $\mathcal{H}$. The transformation e^{iQ} transforms every basis vector $\varphi_{\{n_k\}}$ into one $\varphi^{\prime}_{\{n^{\prime}_k\}} = e^{iQ}\varphi_{\{n_k\}}$ of $\mathcal{H}^{'}$, given by the formula

$$\varphi^{\prime}_{\{n^{\prime}_k\}} = \lim_{N\to\infty} \prod_{k=-N/2}^{N/2} \left\{ \left[\delta_{n_k,1} - \left(a^{\dagger}_{k,+} a^{\dagger}_{-k,-}\right)^{n_k+1}\right] \sin\alpha_k + \left(a^{\dagger}_{k,+} a^{\dagger}_{-k,-}\right)^{n_k} \cos\alpha_k \right\} \phi_0 \ . \tag{2.11}$$

The result is that the scalar product $\left(\psi, e^{iQ}\varphi_{\{n_k\}}\right)$ for every basis vector ψ of $\mathcal{H}$ given by (2.3) is either identically equal to zero or equal to the infinite product

$$\left(\psi_{\{n^{\prime}_{k^{\prime}}\}}, e^{iQ}\varphi_{\{n_k\}}\right) = \lim_{N\to\infty} \prod_{k^{\prime},k=-N/2}^{N/2} \left(S_{k,k^{\prime}} \sin\alpha_k + C_{k,k^{\prime}} \cos\alpha_k\right) , \tag{2.12}$$

where

$$\begin{aligned} S_{k,k^{\prime}} &= \delta_{0,n^{\prime}_{k^{\prime}}} \delta_{1,n_k} - \delta_{0,n_k}\, \delta_{1,n^{\prime}_{k^{\prime}}} \delta_{k,k^{\prime}} \\ C_{k,k^{\prime}} &= \delta_{n_k,n^{\prime}_{k^{\prime}}} \delta_{k,k^{\prime}} \ . \end{aligned} \tag{2.13}$$

However, the infinite product (2.12) diverges also to zero in the limit $N \to \infty$. This is so because by noting the properties of the coefficients $S_{k,k^{\prime}}$ and $C_{k,k^{\prime}}$ given by (2.13) we see that the scalar product (2.12) reduces to the infinite product of the following type

$$\left(\psi_{\{n^{\prime}_{k^{\prime}}\}}, e^{iQ}\varphi_{\{n_k\}}\right) = \lim_{N\to\infty} \prod_{k^{\prime},k=-N/2}^{N/2} \sin\alpha_{k^{\prime}}\ \cos\alpha_k = 0 \ , \tag{2.14}$$

The last relation implies that the Hilbert space $\mathcal{H}^{'}$ spanned by the transformed state vectors $\psi^{'} = e^{iQ}\psi$ contains a subspace of the state vectors $\varphi^{\prime}_{\{n_k\}}$ which are orthogonal to every vector ψ of $\mathcal{H}$. We make the same conclusion as in Haag's work [1]: $c_{k,\sigma}$ and $c^{\dagger}_{k,\sigma}$ given by (2.4) are operators satisfying the same canonical ring as (2.1), i.e.,

$$\left\{c_{k,\sigma}, c^{\dagger}_{k^{\prime},\sigma^{\prime}}\right\} = \delta_{k,k^{\prime}} \delta_{\sigma,\sigma^{\prime}} \ , \ \left\{c_{k^{\prime},\sigma^{\prime}}, c_{k,\sigma}\right\} = \left\{c^{\dagger}_{k^{\prime},\sigma^{\prime}}, c^{\dagger}_{k,\sigma}\right\} = 0 \tag{2.15}$$

but there is no proper unitary transformation connecting these two operator systems. The operators $\left(a_{k,\sigma}, a^{\dagger}_{k,\sigma}\right)$ and the operators $\left(c_{k,\sigma}, c^{\dagger}_{k,\sigma}\right)$ act in two different Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^{'}$ respectively. In Haag's terminology, they belong to inequivalent representations of the same anticommutator ring (2.1) or (2.15) of the field operators. It is a simple matter to show that among all transformed vectors $\psi^{'} = e^{iQ}\psi$ there is no one such as $\phi^{'}_0$ for which

$$c_{k,\sigma}\phi^{'}_0 = 0 \tag{2.16}$$

for all (k, σ). Next we supplement the new Hilbert space $\mathcal{H}^{'}$ spanned by all state vectors $\psi^{'} = e^{iQ}\psi$ with the vacuum state $\phi^{'}_0$ satisfying the auxiliary condition (2.16).

Each inequivalent representation of the anticommutator ring (2.15) with the subsidiary condition (2.16) is specified by a chosen infinite set of parameters α_k entering the transformations (2.4)-(2.9). Thus, the number of the inequivalent representations of (2.15) is indeed infinite. In contradistinction to inequivalent representations of Lie algebras, where the representations are specified by a finite number of Casimir operator eigenvalues, the inequivalent representations of

the anticommutator ring (2.15) of the field operators $c_{k,\sigma}$ and $c^{\dagger}_{k,\sigma}$ are specified by infinite set of parameters.

Our analysis of inequivalent representations of the anticommutator ring of the field operators (2.1), as presented above, can be generalized in a straightforward way to any set of quantum numbers (k, σ) and to many other classes of inequivalent representations both for fermions and bosons.

Next we consider a Hamiltonian H governing a physical system with an infinite number of degrees of freedom as a given function of the operators $a_{k,\sigma}$ and $a^{\dagger}_{k,\sigma}$, i.e.,

$$H = H(a^{\dagger}, a) \ . \tag{2.17}$$

The field operators $a_{k,\sigma}$ and $a^{\dagger}_{k,\sigma}$ and the corresponding Hamiltonian (2.17) act in the representation space which is the Hilbert space $\mathcal{H}$ spanned by the basis vectors given by (2.3). In this representation we have the corresponding partition function

$$Z = Tr e^{-\beta H} \tag{2.18}$$

and statistical average values $\big\langle A \big\rangle$ of physical observables associated with operators $A(a^{\dagger}, a)$ given by the relations

$$\big\langle A \big\rangle = \frac{1}{Z} Tr \big\{ A(a^{\dagger}, a) e^{-\beta H} \big\} \ . \tag{2.19}$$

For each inequivalent representation of the anticommutator ring (2.15) we can construct the transformed Hamiltonian

$$\widetilde{H}(c^{\dagger}, c) = e^{iQ} H(a^{\dagger}, a) e^{-iQ} \tag{2.20}$$

in its normal form by employing the canonical anticommutator relations (2.15). The corresponding partition function

$$\widetilde{Z} = Tr e^{-\beta \widetilde{H}(c^{\dagger}, c)} \tag{2.21}$$

can be different from that given by (2.18) because the operators $H(a^{\dagger}, a)$ and $\widetilde{H}(c^{\dagger}, c)$ act in two different Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^{'}$ respectively. By the same way, the statistical average value $\big\langle A \big\rangle$ corresponding to a physical observable

$$\big\langle A \big\rangle = \frac{1}{\widetilde{Z}} Tr \big\{ \widetilde{A}(c^{\dagger}, c) e^{-\beta \widetilde{H}} \big\} \ . \tag{2.22}$$

can be different from that given by (2.19). These conclusions may seem to be paradoxes residing in facts as if physical observables were dependent on our will how we select a single inequivalent representation for the anticommutator ring of the field operators (2.15). There is, in fact, no freedom in selecting an appropriate inequivalent representation of (2.15). As we have mentioned in the introduction, the second law of thermodynamics dictates uniquely which inequivalent representation is relevant for describing the physical system at given values of thermodynamic variables.

One may even incorrectly believe that the conclusion concerning the different results for the same physical quantities given by the relations (2.18) and (2.21) or by the relations (2.19) and (2.22) are wrong. Such an incorrect belief is supported by the formal appearance of the transformation (2.20) which seems to be a similarity transformation. The unpermited application of the cyclic properties for the trace of infinitely dimensional matrices product would lead to incorrect conclusions that the results of the relations (2.18), (2.20) or (2.19), (2.21) are identical.

We conclude this section by stating that in quantum field theory a given Hamiltonian H of a system leads to distinct results for the partition function Z and for the statistical average values of physical observables $\big\langle A \big\rangle$ depending on a selected inequivalent representation of the anticommutator ring of field operators. These conclusions will be explicitly demonstrated on the BCS model Hamiltonian of superconductivity.

3. One method for the selection of individual inequivalent representations

In this section we elucidate how one tacitly selects a single inequivalent representation out of infinitely many representations of the commutator or anticommutator ring (1.1) of field operators in a practical application of quantum field theory. In quantum field theories with interactions between fields there is not known even one physical example with an exact solution. In all practical applications one divides Hamiltonian H of a system into the sum

$$H = H_0(a^\dagger, a) + H_I(a^\dagger, a) \tag{3.1}$$

where H_0 is called the unperturbed Hamiltonian and the remaining term H_I is called the perturbative part. The unperturbed Hamiltonian H_0 is chosen in a way to be exactly diagonalized and by this fact its effects are treated exactly. Its eigenvalues ψ_μ, where μ denotes an array with an infinite number of items form a complete basis of a Hilbert space $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is the representation space for a single inequivalent representation of the commutator or anticommutator ring (2.1) of the field operators $a_{k,\sigma}$ and $a^\dagger_{k,\sigma}$ entering the Hamiltonian (3.1). The unperturbed partition function

$$Z_0 = Tre^{-\beta H_0(a^\dagger, a)} \tag{3.2}$$

can be exactly evaluated and is typical for the chosen inequivalent representation. The total partition function Z is expressed by the perturbation series

$$Z = Tre^{-\beta H} = Z_0 \Big\langle T \exp\Big\{ - \int_0^\beta d\tau V(\tau) \Big\} \Big\rangle_0 = Z_0 \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!} \Big\langle T \Big(\int_0^\beta d\tau V(\tau) \Big)^\nu \Big\rangle_0 \tag{3.3}$$

where the symbol T stands for the time-ordered product,

$$V(\tau) = e^{\tau H_0} H_I e^{-\tau H_0} \tag{3.4}$$

and

$$\Big\langle T \Big(\int_0^\beta d\tau V(\tau) \Big)^\nu \Big\rangle_0 \tag{3.5}$$

denotes the statistical average value of the operator inside the brackets $\langle ... \rangle_0$ with respect to the unperturbed Hamiltonian H_0. It is needless to say that the statistical average values (3.5) are evaluated in the chosen inequivalent representation.

However, the splitting of the total Hamiltonian as given by (3.1) is not unique. One can equally well divide the same Hamiltonian as

$$H = H_0^{'}(a^\dagger, a) + H_I^{'}(a^\dagger, a) \tag{3.6}$$

where the new unperturbed Hamiltonian $H_0^{'}$ is by definition different from H_0 and is not related to H_0 by any proper unitary transformation. The new unperturbed Hamiltonian $H_0^{'}$ is assumed to be diagonalized by such transformations of the field operators $a_{k,\sigma}$ and $a^\dagger_{k,\sigma}$ like (2.4) in order to achieve its diagonal form

$$\widetilde{H}_0(c^\dagger, c) = e^{iQ} H_0^{'}(a^\dagger, a) e^{-iQ} \ . \tag{3.7}$$

The requirement to have $\widetilde{H}_0(c^\dagger, c)$ in its diagonal form puts certain constraints on the transformation parameters α_k entering the transformations (2.4)-(2.9). In other words, all transformation parameter α_k values are determined by a finite set of physical parameters present in the chosen unperturbed Hamiltonian $H_0^{'}(a^\dagger, a)$. Since the infinite set of parameters α_k specifies a single inequivalent representation of the commutator or anticommutator ring of the field operators (2.15), with new chosen unperturbed Hamiltonian $H_0^{'}$ one again tacitly selects another single inequivalent representation. The eigenvalues ψ_μ' of $\widetilde{H}_0$ form again a complete basis of a new Hilbert space $\mathcal{H}^{'}$ for the new selected inequivalent representation.

In this inequivalent representation one gets the unperturbed partition function

$$\widetilde{Z}_0 = Tre^{-\beta \widetilde{H}_0(c^\dagger, c)} \tag{3.8}$$

which is, of course, different from (3.2) by the definitions (3.1) and (3.6). The corresponding total partition function

$$\widetilde{Z} = Tre^{-\beta \widetilde{H}(c^{\dagger},c)} = \widetilde{Z}_0 \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \left\langle T \left(\int_0^{\beta} d\tau \widetilde{V}(\tau) \right)^{\nu} \right\rangle_0 \tag{3.9}$$

is also distinct from that given by (3.3) because the same Hamiltonian H has two different, so to speak, "matrix" realizations (3.1) and (3.3) corresponding to two different inequivalent representations of the anticommutator ring of field operators (2.1) or (2.15).

By the same approach as outlined above, one can continue to study a series of inequivalent representations associated with a given Hamiltonian H. By selecting a series $H_0 = H_{01}, H_{02}, H_{03}, ...$ of unperturbed Hamiltonians H_{01}, H_{02}, $H_{03}, ...$ one can explore physical properties of the corresponding series of different Gibbs states associated with the same Hamiltonian H.

The aforementioned methods will be explicitly demonstrated for the investigation of physical properties of several distinct Gibbs states associated with the BCS model Hamiltonian in theory of superconductivity [2]. The BCS Hamiltonian has been deliberately chosen from three different reasons.

At first, the BCS Hamiltonian is generally very well-known, extraordinary simple and despite of its simplicity it has been very successful in explaining properties of a large class of the so-called low temperature superconductors (LTS) on terms of only two phenomenological material parameters. On the other hand, properties of high temperature superconductors (HTS) having anisotropic layered structures are claimed to be unexplainable on the basis of the BCS theory.

At second, the BCS theory as a theory with a nontrivial interactions between electrons has provided us with two different solutions for the Gibbs states which are asymptotically exact in the thermodynamic limit.

At third, the BCS theory has been studied from various aspects for many years. That is why there is a generally spread belief that its consequences are completely exhausted and therefore there are no motivations for its further exploration. However, its consequences has not been studied from the point of view of inequivalent representations of the anticommutator ring of electron field operators (2.1) or (2.15) yet. Since the ring has infinitely many inequivalent representations the BCS Hamiltonian should have infinitely many Gibbs states. It is, therefore, worthwhile to undertake the task to find at least one new Gibbs state associated with the BCS Hamiltonian.

4. The applications on the BCS theory of superconductivity

The BCS theory is based on the following well-known grand canonical Hamiltonian

$$H = \sum_{\vec{k},\sigma} \xi_{\vec{k}} a^\dagger_{\vec{k},\sigma} a_{\vec{k},\sigma} - \frac{g}{V} \sum_{\vec{k},\vec{k}'} a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} a_{-\vec{k}',-} a_{\vec{k}',+} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid)\Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \equiv$$

$$\equiv K + H_I \ , \tag{4.1}$$

where $\xi_{\vec{k}} = \frac{\hbar^2}{2m}\vec{k}^2 - \mu$ is the kinetic energy of an electron (in the state specified by the wave vector $\vec{k}$) counted from the chemical potential μ which can be approximated by the Fermi energy ϵ_F, i.e., $\mu \doteq \epsilon_F$. The index $\sigma = \pm$ denotes the spin 1/2 projection of an electron, the symbols $a_{\vec{k},\sigma}$ and $a^\dagger_{\vec{k},\sigma}$ are annihilation and creation operators of electrons in the states $\left(\vec{k},\sigma\right)$ resp. Finally, g is the squared electron-phonon coupling constant, ω_D is the Debye frequency, V is the volume of the system. K denotes the electron kinetic energy operator and H_I is the interaction term. The sum over $\vec{k}$ and $\vec{k}^{'}$ in H_I is restricted by the conditions $\mid \xi_{\vec{k}} \mid < \hbar\omega_D$, $\mid \xi_{\vec{k}'} \mid < \hbar\omega_D$, as indicated by the appropriate step functions $\Theta(x)$ in the relation (4.1).

All thermodynamic properties of the system governed by the Hamiltonian (4.1) are determined by the grand canonical partition function

$$Z = Tre^{-\beta H} \tag{4.2}$$

The grand canonical potential

$$\Omega = -k_B T \ln Z \tag{4.3}$$

specifies the Gibbs state of the system and provides us with complete information on thermodynamic properties of the system described by the Hamiltonian (4.1). Ω is a function of the thermodynamic variables T, V and μ, and in addition to them, it is also a function of the coupling constant g, i.e., $\Omega \equiv \Omega(T,V,\mu;g)$. From the definitions (4.2) and (4.3) it follows

$$\frac{\partial\Omega}{\partial g} = \frac{1}{g}\langle H_I \rangle \ , \tag{4.4}$$

where the statistical average $\langle H_I \rangle$ is, of course, a non trivial function of g, i.e., $\langle H_I \rangle = \langle H_I \rangle(g)$. Suppose that one has calculated $\langle H_I \rangle(g)$, then he can integrate the equation (4.4) to get the exact relation

$$\Omega - \Omega_n = \int_0^g \frac{dg'}{g'} \langle H_I \rangle(g') \ , \tag{4.5}$$

where $\Omega \equiv \Omega(T,V,\mu;g)$ is the grand canonical potential as defined by (4.2) and (4.3) and $\Omega_n = \Omega(T,V,\mu;g=0)$ is the grand canonical potential corresponding to the ideal electron gas. The grand canonical potential difference (4.5) is particularly convenient for two reasons. First, it expresses physical anomalies associated with the interaction term H_I above the background corresponding to properties of the ideal electron gas. Second, it exhibits manifestly the dependence of Ω on a chosen inequivalent representation of the anticommutator ring (2.1) of the electron field operators entering the Hamiltonian H. In practical calculations one cannot evaluate the exact average value $\langle H_I \rangle$ and must resort to a perturbation theory by selecting an unperturbed Hamiltonian H_0 which can be exactly diagonalized.

With a chosen unperturbed Hamiltonian H_0 one calculates the unperturbed grand canonical partition function (3.2) and the corresponding statistical averages $\langle A \rangle_0$ of operators A defined by the relations

$$\langle A \rangle_0 = \frac{1}{Z_0} Tr\left(Ae^{-\beta H_0}\right) \ . \tag{4.6}$$

The average value $\langle H_I \rangle(g)$ entering the relations (4.4) and (4.5) can be expressed in the form

$$\langle H_I \rangle(g) = \langle H_I \rangle_0(g) + R(g) \ , \tag{4.7}$$

where $R(g)$ represents symbolically all contributions coming from remaining perturbative terms.

First we choose H_0 to be the electron kinetic energy operator in (4.1), i.e.,

$$H_{01} = \sum_{\vec{k},\sigma} \xi_{\vec{k}} a^\dagger_{\vec{k},\sigma} a_{\vec{k},\sigma} \ . \tag{4.8}$$

In this selection, the unperturbed Hamiltonian H_{01} is already diagonalized and the complete operator system is given by the electron annihilation and creation operators $a_{\vec{k},\sigma}$ and $a^\dagger_{\vec{k},\sigma}$ resp. in the quantum states $\big(\vec{k},\sigma\big)$. These operators obey the canonical anticommutator ring (2.1) with the subsidiary condition (2.2). In this case the Hilbert space $\mathcal{H}_1$ is the space of a single inequivalent representation of the anticommutator ring (2.1) with the auxiliary condition (2.2). Each basis vector ψ is then specified by an infinite array $\big\{n_{\vec{k},\sigma}\big\}$ of the occupation numbers $n_{\vec{k},\sigma} = 0, 1$ for each one particle state $\big(\vec{k},\sigma\big)$, i.e.,

$$\psi\big(\{n_{\vec{k},\sigma}\}\big) = \prod_{\vec{k},\sigma} \big(a^\dagger_{\vec{k},\sigma}\big)^{n_{\vec{k},\sigma}} \phi_0 \ . \tag{4.9}$$

With this choice of the unperturbed Hamiltonian H_{01} one gets the spectrum $E_{\vec{k},\sigma}$ of elementary excitations

$$E_{\vec{k},\sigma} = \xi_{\vec{k}} \ , \ \xi_{\vec{k}} \in < -\mu, \infty) \tag{4.10}$$

and the statistical average values

$$\big\langle a_{\vec{k}',\sigma'}, a_{\vec{k},\sigma} \big\rangle = \big\langle a^\dagger_{\vec{k}',\sigma'}, a^\dagger_{\vec{k},\sigma} \big\rangle = 0 \tag{4.11}$$

valid in all orders of the perturbation theory. The statistical average values valid in the first order of the perturbation theory have the following forms

$$\big\langle a^\dagger_{\vec{k}',\sigma'}, a_{\vec{k},\sigma} \big\rangle_{01} = \frac{\delta_{\vec{k},\vec{k}'} \delta_{\sigma,\sigma'}}{e^{\beta \xi_{\vec{k}}} + 1} \tag{4.12}$$

$$\begin{aligned} \big\langle H_I \big\rangle_{01} &= -\frac{g}{V} \sum_{\vec{k}} \frac{\Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid)}{(e^{\beta \xi_{\vec{k}}} + 1)^2} = \\ &= -g N(0) k_B T \left(\ln \left[\frac{1 + \tanh \frac{\beta\hbar\omega_D}{2}}{1 - \tanh \frac{\beta\hbar\omega_D}{2}} \right] - \tanh \frac{\beta\hbar\omega_D}{2} \right) , \end{aligned} \tag{4.13}$$

where

$$N(0) = \frac{m k_F}{2\pi^2 \hbar^2} \tag{4.14}$$

is the density of electron states for one spin projection at the Fermi surface. In evaluating $\big\langle H_I \big\rangle_{01}$ the relations $\mu \approx \epsilon_F$ and $\hbar\omega_D \ll \epsilon_F$ have been used. From the relation (4.13) one sees that $\big\langle H_I \big\rangle_{01}$ does not scale with the volume V and the same is true for all remaining terms $R(g)$ in (4.7), as was formally proven in [3] and demonstrated by explicit calculations in [4]. The result (4.13) together with (4.7) implies the relation

$$\Omega - \Omega_n = \big\langle H_I \big\rangle_{01} + R(g) \ . \tag{4.15}$$

The right-hand side of the last equation is not proportional to the volume V and by this fact the density of the grand canonical potential $\frac{\Omega}{V}$ approaches the density $\frac{\Omega_n}{V}$ corresponding to the ideal electron gas in the thermodynamic limit $V \to \infty$. In other words, the interaction term H_I in (4.1) has no macroscopic effects on $\Omega(T, V, \mu; g)$ in the thermodynamic limit $V \to \infty$ provided that one has chosen the unperturbed Hamiltonian $H_0 = H_{01}$ as given by (4.8), i.e.,

$$\Omega(T, V, \mu; g) = \Omega_n(T, V, \mu) \tag{4.16}$$

for the chosen inequivalent representation in the thermodynamic limit.

Next we choose the unperturbed Hamiltonian H_{02}, to be the one corresponding to the standard BCS theory of superconductivity, i.e. in the form

$$H_{02} = \frac{V}{g}\Delta^*\Delta + \sum_{\vec{k},\sigma} \xi_{\vec{k}} a^\dagger_{\vec{k},\sigma} a_{\vec{k},\sigma} - \sum_{\vec{k}} \left(\Delta a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} + \Delta^* a_{-\vec{k},-} a_{\vec{k},+} \right) \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; , \tag{4.17}$$

where Δ and Δ^* are the following average values

$$\Delta = \frac{g}{V} \sum_{\vec{k}} \langle a_{-\vec{k},-} a_{\vec{k},+} \rangle_{02} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; ,$$
$$\Delta^* = \frac{g}{V} \sum_{\vec{k}} \langle a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} \rangle_{02} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{4.18}$$

called the gap functions. H_{02} is the Hamiltonian (4.1) in the so called mean field approximation.

The unperturbed Hamiltonian H_{02} is diagonalized by means of transformations (2.8), (2.9) with the transformation parameters $\alpha_{\vec{k}}$ determined by the formula

$$\sin^2 \alpha_{\vec{k}} = \frac{1}{2}\left(1 - \frac{\xi_{\vec{k}}}{\sqrt{\xi^2_{\vec{k}} + \Delta^2}} \right) \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \;\; , \Delta = \Delta^* \; . \tag{4.19}$$

All the transformation parameters $\alpha_{\vec{k}}$ are determined in terms of the physical parameters entering H_{02} and specify the single inequivalent representation of the anticommutator ring (2.15) with the auxiliary condition (2.16). Its representation space is the Hilbert space denoted by $\mathcal{H}_2$. The unperturbed Hamiltonian H_{02} has the following diagonal form

$$\widetilde{H}_{02} = \frac{V}{g}\Delta^2 + \sum_{\vec{k},\sigma} \left[E_{\vec{k}} c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{1}{2}(\xi_{\vec{k}} - E_{\vec{k}}) \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \right] \; , \tag{4.20}$$

where $E_{\vec{k}}$ is the energy spectrum of elementary excitations given by the formulae

$$E_{\vec{k}} = \sqrt{\Delta^2 + \xi^2_{\vec{k}}}\; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; ,$$
$$E_{\vec{k}} = \xi_{\vec{k}}\; \Theta(\mid \xi_{\vec{k}} \mid - \hbar\omega_D) \;\; ; \;\; \xi_{\vec{k}} \in < -\mu, +\infty > \;\; . \tag{4.21}$$

The energy spectrum $E_{\vec{k}}$ as the function of ξ is schematically depicted on Fig. 1.

As is seen from the Fig. 1 the energy spectrum $E_{\vec{k}}$ has discontinuities at two fixed points $\xi = \pm\hbar\omega_D$ which are specified by the material parameter ω_D.

The unperturbed Hamiltonian H_{02} as given by (4.17) or (4.20) both determines and is determined by the average values (4.18). Thus the relation (4.18) is, in fact, the self-consistency condition known in all text books [6] as the gap equation

$$\Delta = g\frac{\Delta}{(2\pi)^3} \int \frac{d^3\vec{k}}{2E_{\vec{k}}} \tanh \frac{1}{2}\beta E_{\vec{k}}\; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; . \tag{4.22}$$

The gap function Δ is different from zero, $\Delta \neq 0$, for $T < T_c$, where T_c is the critical temperature given by the formula [6]

$$T_c = \frac{2e^\gamma}{\pi} T_D \exp\left[-\frac{1}{gN(0)} \right] \doteq 1.13\; T_D \exp\left[-\frac{1}{gN(0)} \right] \; . \tag{4.23}$$

Here $\gamma \doteq 1.781$ is Euler constant and $T_D = \frac{\hbar\omega_D}{k_B}$ is the Debye temperature. The critical temperature T_c, as seen from (4.23), is proportional to T_D and is a singular function of g at $g = 0$.

The solution to the gap equation (4.22) provides $\Delta = \Delta(T, g)$ as a function of T and g, which cannot be expressed in an analytic form, but only numerically. The only contribution to $\langle H_I \rangle(g)$ which survives the thermodynamic limit $V \to \infty$ has the form

$$\langle H_I \rangle(g) = \langle H_I \rangle_{02}(g) = -\frac{V}{g}\Delta^2(g) \tag{4.24}$$

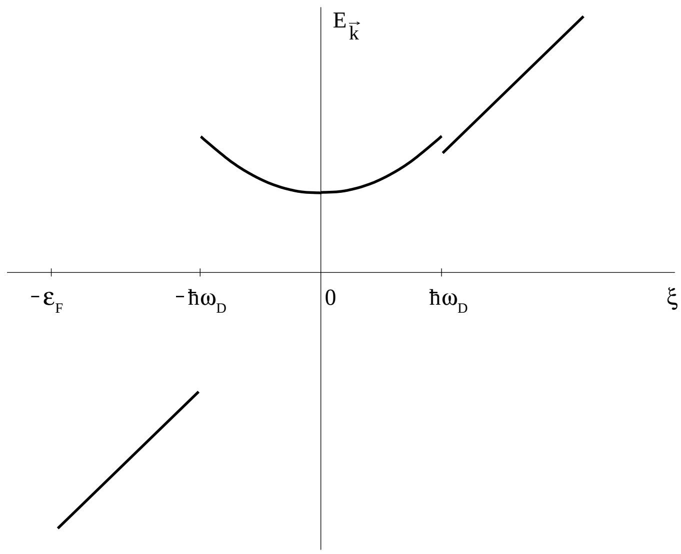

FIGURE 1. The energy spectrum $E_{\vec{k}}$ as function of ξ.

because all remaining terms denoted by $R(g)$ in (4.7) become negligible in this limit as formally proven in [3] and demonstrated by explicit calculations in [4].

Two different results (4.13) and (4.24) for the average value of the same physical observable $\langle H_I \rangle$ exhibit clearly and evidently the important role of the inequivalent representations in practical applications. Namely, $\langle H_I \rangle$ has no macroscopic effects if evaluated in the representation Hilbert space $\mathcal{H}_1$, however, $\langle H_I \rangle$ has a very relevant macroscopic contribution if evaluated in the representation Hilbert space $\mathcal{H}_2$.

The result (4.24) with the relation (4.5) gives the expression for the grand canonical potential $\Omega(T, V, \mu; g)$ in the form

$$\Omega(T, V, \mu; g) = \Omega_n(T, V, \mu) - \frac{V}{8\pi} H_c^2(T) \,, \tag{4.25}$$

where $H_c(T)$ is the thermodynamic critical magnetic field defined by the relation

$$H_c^2(T) = 8\pi \int_0^g \frac{dg'}{g'^2}\, \Delta^2(g') \,. \tag{4.26}$$

The solution to the BCS Hamiltonian (4.1), as represented by the relations (4.20)-(4.26), describes, as is well-known, the superconducting state of the system governed by the Hamiltonian (4.1). This solution will be referred as to the standard solution of the BCS theory of superconductivity. We have discussed it briefly in order to demonstrate the role of inequivalent representations of the canonical anticommutator ring (2.1) or (2.15) of electron field operators on solutions which are generally very well-known.

The main purpose of this section is to explore the physical implications of an additional inequivalent representation of the anticommutator ring (2.15) associated with the Hamiltonian (4.1) of the BCS theory. Without any physical motivation we choose the third form of the unperturbed Hamiltonian H_{03}, as given by

$$H_{03} = \frac{g}{V} \sum_{\vec{k},\vec{k}'} \langle a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} \rangle_{03} \langle a_{-\vec{k}',-} a_{\vec{k}',+} \rangle_{03} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid)\, \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid)\, \Theta(q - \mid k - k' \mid) +$$
$$+ \sum_{\vec{k},\sigma} \xi_{\vec{k}} a^\dagger_{\vec{k},\sigma} a_{\vec{k},\sigma} - \sum_{\vec{k}} \left(\Delta_{\vec{k}} a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} + \Delta^*_{\vec{k}} a_{-\vec{k},-} a_{\vec{k},+} \right) \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \,, \tag{4.27}$$

where

$$\Delta_{\vec{k}} = \frac{g}{V} \sum_{\vec{k}'} \langle a_{-\vec{k}',-} a_{\vec{k}',+} \rangle_{03} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \, \Theta(q - \mid k - k^{'} \mid) \, ,$$

$$\Delta^*_{\vec{k}} = \frac{g}{V} \sum_{\vec{k}'} \langle a^\dagger_{\vec{k}',+} a^\dagger_{-\vec{k}',-} \rangle_{03} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \, \Theta(q - \mid k - k^{'} \mid) \tag{4.28}$$

are gap functions which are dependent on the wave vector $\vec{k}$. Here, q is chosen to be the absolute value of the wave vector corresponding to the minimal energy of an electron confined in a box with the edges L_1, L_2 and L_3, i.e.,

$$\frac{\hbar^2 q^2}{2m} = \frac{\hbar^2}{2m} \left[\left(\frac{\pi}{L_1} \right)^2 + \left(\frac{\pi}{L_2} \right)^2 + \left(\frac{\pi}{L_3} \right)^2 \right] . \tag{4.29}$$

One may say that H_{03} is a kind of an interpolation between the Hamiltonians H_{01} and H_{02}. Indeed, H_{03} contains only those terms from H_{02} which partially conserve the energy in microscopic scattering processes between electrons. This fact is represented by the presence of the step function $\Theta(q - \mid k - k^{'} \mid)$ in (4.27) and (4.28). One may regard H_{03} as a mathematical toy in order to search for an additional inequivalent representation of the anticommutator ring (2.15) within the framework of the same Hamiltonian (4.1).

The Hamiltonian H_{03} is again diagonalized by the well-known Bogoliubov-Valatin transformations (2.8)-(2.9). The parameters $\alpha_{\vec{k}}$ of the transformations (2.4)-(2.9) are defined by the relation

$$\sin \alpha_{\vec{k}} = -\frac{1}{\sqrt{2}} \left(1 - \frac{\xi_{\vec{k}}}{\sqrt{\xi^2_{\vec{k}} + \mid \Delta_{\vec{k}} \mid^2}} \right)^{\frac{1}{2}} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{4.30}$$

with the gap functions $\Delta_{\vec{k}}$ depending on $\vec{k}$, which specify the chosen inequivalent representation of the anticommutator ring (2.15). In this representation the unperturbed Hamiltonian H_{03} gets its diagonal form

$$\widetilde{H}_{03} = \frac{g}{V} \sum_{\vec{k},\vec{k}'} \langle a^\dagger_{\vec{k},+} a^\dagger_{-\vec{k},-} \rangle_{03} \langle a_{-\vec{k}',-} a_{\vec{k}',+} \rangle_{03} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \, \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \, \Theta(q - \mid k - k^{'} \mid) +$$

$$+ \sum_{\vec{k},\sigma} \left[E_{\vec{k}} c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{1}{2} (\xi_{\vec{k}} - E_{\vec{k}}) \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \right] , \tag{4.31}$$

where

$$E_{\vec{k}} = \sqrt{\mid \Delta_{\vec{k}} \mid^2 + \xi^2_{\vec{k}}} \, \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \, ,$$

$$E_{\vec{k}} = \xi_{\vec{k}} \, \Theta(\mid \xi_{\vec{k}} \mid - \hbar\omega_D) \;\; ; \; \xi_{\vec{k}} \in (-\mu, +\infty) \tag{4.32}$$

is the energy spectrum of quasiparticles represented by the field operators $c_{\vec{k},\sigma}$ and $c^\dagger_{\vec{k},\sigma}$. By evaluating the statistical average value $\langle a_{-\vec{k},-} a_{\vec{k},+} \rangle_{03}$ we get the following relation

$$\langle a_{-\vec{k},-} a_{\vec{k},+} \rangle_{03} = \frac{1}{2E_{\vec{k}}} \Delta_{\vec{k}} \tanh \frac{\beta E_{\vec{k}}}{2} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \, . \tag{4.33}$$

The last relation, when inserted into the definitions (4.28), gives us the self-consistency condition for the gap function

$$\Delta_{\vec{k}} = \frac{g}{V} \sum_{\vec{k}'} \frac{\Delta_{\vec{k}'}}{2E_{\vec{k}'}} \tanh \frac{\beta E_{\vec{k}'}}{2} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \, \Theta(q - \mid k - k^{'} \mid) \, . \tag{4.34}$$

The equations (4.33) and (4.34) determining $\langle a_{-\vec{k},-} a_{\vec{k},+} \rangle_{03}$ and the gap function $\Delta_{\vec{k}}$ resp. are, of course, closely related to those ones in the standard solution to the BCS theory of the superconductivity except for the $\vec{k}$ dependence of $\Delta_{\vec{k}}$ and the presence of the step function $\Theta(q - \mid k - k^{'} \mid)$

in (4.34) by which a partial energy conservation rule is required even for microscopic scattering processes between electrons.

5. The solution to the gap equation

The solution to the gap equation (4.34) is analyzed similarly as in the work [7]. The assumption that the gap function $\Delta_{\vec{k}}$ is a function of the magnitude $k =| \vec{k} |$ only, simplifies the relation (4.34) substantially. We employ the inequality $q \ll k^{'}$ which is valid for every vector $\vec{k}^{''}$ in the sum (4.34). By this fact, we can replace the equation (4.34) by the relation

$$\Delta_{\vec{k}} = \frac{g}{2V} \frac{\nu(k)}{E_{\vec{k}}} \Delta_{\vec{k}} \tanh \frac{\beta E_{\vec{k}}}{2} \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{5.1}$$

where

$$\nu(k) = \sum_{\vec{k}'} \Theta(q - \mid k - k^{'} \mid) \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}'} \mid) \; . \tag{5.2}$$

We emphasize that the equation (5.1) takes into account only linear terms in the tiny quantity q. The quantity $\nu(k)$, as given by the equation (5.2), is, in fact, the number of wave vectors $\vec{k}^{''}$ with the lengths $k^{'}$ from the interval $k - q < k^{'} < k + q$. In order to evaluate the number $\nu(k)$ we remind ourselves that the vector $\vec{k}^{''}$ have the components given below

$$\vec{k}^{''} = \left(\frac{2\pi}{L_1} n_1, \frac{2\pi}{L_2} n_2, \frac{2\pi}{L_3} n_3 \right) , \tag{5.3}$$

where n_1, n_2, n_3 are integers and L_1, L_2, L_3 are edge lengths of a box into which the electron system is enclosed. The given magnitude $k^{'}$ defines the surface of an ellipsoid with semiaxes $a_1 = k^{'} \frac{L_1}{2\pi}$, $a_2 = k^{'} \frac{L_2}{2\pi}$ and $a_3 = k^{'} \frac{L_3}{2\pi}$ in an Euclidean space with coordinates n_1, n_2 and n_3, i.e.,

$$\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} = 1 \; . \tag{5.4}$$

Thus the number of the states $\nu(k)$ is approximately equal to the volume enclosed between two surface areas of the ellipsoids corresponding to the minimal and maximal semiaxes a_1, a_2 and a_3, i.e.,

$$\nu(k) = \frac{1}{\pi^2} \, q L_1 L_2 L_3 k^2 \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; . \tag{5.5}$$

We analyze the number $\nu(k)$ for two limiting cases, namely, when the ellipsoid (5.4) degenerates either into a sphere or into a thin circular disc. The sphere corresponds to the case when the electron system under consideration is enclosed in a cube with the edges $L_1 = L_2 = L_3 = L$, i.e., it has isotropic bulk properties. If the ellipsoid (5.4) degenerates into a thin circular disc with $L_1 = L_2 \gg L_3 = d$, then the system of electrons has a form of a thin film of the given thickness d, i.e., it has anisotropic properties. Both of these limiting cases represent objects of the physical interest.

We first analyze the case corresponding to the isotropic bulk material. In this case we get

$$\nu(k) = \frac{2\sqrt{3}}{\pi} \, k^2 L^2 \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; . \tag{5.6}$$

By inserting the last result with $V = L^3$ into the equation (5.1) we get the relation

$$\Delta_{\vec{k}} = \frac{\sqrt{3} g}{\pi L} \, \Delta_{\vec{k}} \, \frac{k^2}{E_{\vec{k}}} \, \tanh \frac{\beta E_{\vec{k}}}{2} \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{5.7}$$

which in the thermodynamic limit $L \to \infty$ has the only solution $\Delta_{\vec{k}} = 0$. Thus this type of superconductivity cannot exist in three dimensional isotropic materials.

We next analyze the second limiting case when the system has a form of a thin film with a given thickness d. In this case $V = L^2 d$, $q \doteq \frac{\pi}{d}$ and the number of the states $\nu(k)$ is given by the relation

$$\nu(k) = \frac{k^2 L^2}{\pi} \; \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \; . \tag{5.8}$$

By inserting the last result into equation (5.1) we get the relation

$$\Delta_{\vec{k}} = \frac{g}{2\pi d} \frac{k^2}{E_{\vec{k}}} \Delta_{\vec{k}} \tanh \frac{\beta E_{\vec{k}}}{2} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{5.9}$$

which can have a nontrivial solution $\Delta_{\vec{k}} \neq 0$ in the thermodynamic limit $L \to \infty$.

Since the vectors $\vec{k}$ in the relation (4.28) are restricted by the inequalities below

$$\epsilon_F - \hbar\omega_D \leq \frac{\hbar^2 k^2}{2m} \leq \epsilon_F + \hbar\omega_D \ , \ \hbar\omega_D \ll \epsilon_F \tag{5.10}$$

we can replace the quantity k^2 in (5.9) by k_F^2. Then from the equation (5.9) it follows that the energy spectrum $E_{\vec{k}}$ of the quasiparticles in the superconducting state with $\Delta_{\vec{k}} \neq 0$ must satisfy the following relation

$$1 = G \frac{2\epsilon_F}{E_{\vec{k}}} \ \tanh \frac{\beta E_{\vec{k}}}{2} \Theta(\hbar\omega_D - \mid \xi_{\vec{k}} \mid) \tag{5.11}$$

where

$$G = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{\epsilon_1} \tag{5.12}$$

is the dimensionless effective coupling constant and

$$\epsilon_1 = \frac{\hbar^2}{2m} \frac{\pi^2}{d^2} \tag{5.13}$$

is the minimal energy of an electron confined in a thin layer of the thickness d.

The equation (5.11) determines the quasiparticle energy spectrum $E_{\vec{k}}$ as a certain function of temperature T. It can be satisfied only if the temperature T is below the critical temperature T_c,

$$T_c = G T_F \ , \tag{5.14}$$

where T_F is the Fermi temperature. It is interesting to point out that the relation (5.14) between the critical temperature T_c and the Fermi temperature T_F has accidently the same form as the one found from the analysis of experimental data by Uemura et al. [8] and discussed in [9] on a phenomenological basis. In the phenomenological formula of the form (5.14), proposed in [8] and called as the Uemura plot in [9], the phenomenological parameter G is the same for a unique group of HTS. The universal correlation between T_c and T_F as given by (5.14) has been claimed [8, 9] to exist in all HTS with planar structures. The coincidence between the theoretical formula (5.14) and the phenomenological formula [8, 9] seems to be promising and stimulates us for further theoretical investigation of this type of superconductivity to be compared with the experimental data of planar HTS.

The relation (5.14) determining the critical temperature T_c differs qualitatively from the relation (4.23) corresponding to the standard solution of the BCS theory. T_c is proportional to T_F in contrast to the Debye temperature T_D in (4.23), i.e., T_c does not depend on ionic mass. Thus this type of superconductivity, despite of the fact that it is due to electron-phonon interaction, does not exhibit the isotope effect. T_c is an analytic function of g at $g = 0$. The proportionality coefficient G depends on the lowest bound energy ϵ_1 for the energy spectrum of electrons confined in a layer of the thickness d, i.e., for the thinner layer one has the higher critical temperature T_c. For an isotropic bulk material $\epsilon_1 \to 0$ and in the same way $T_c \to 0$. This property seems to represent the reason why this type of superconductivity cannot exist in isotropic systems, but it can appear in systems with electrons residing on quasi two dimensional planar structures. All these features seem to be promising that this type of superconductivity may have some relevance for the properties of HTS observed in experiments.

The solution to equation (5.11) for the energy spectrum $E_{\vec{k}}$, in the range of $\vec{k}$ for which $\Delta_{\vec{k}} \neq 0$, is a quantity $\epsilon(T)$ which is independent of $\vec{k}$. However, it is a function of T, i.e.,

$$E_{\vec{k}} = \sqrt{\mid \Delta_{\vec{k}} \mid^2 + \xi_{\vec{k}}^2} = \epsilon(T) \ , \ T \leq T_c \ , \ \mid \xi_{\vec{k}} \mid) \leq \epsilon(T) \leq \hbar\omega_D \ ,$$
$$E_{\vec{k}} = \xi_{\vec{k}} \ , \ \mid \xi_{\vec{k}} \mid \geq \epsilon(T) \ , \ \xi_{\vec{k}} \in (-\mu, +\infty) \ . \tag{5.15}$$

One cannot obtain an explicit expression for the solution $\epsilon(T)$ to the equation (5.11) in the form of elementary functions, but has to resort to numerical methods given below. The energy spectrum $E_{\vec{k}}$ as a function of ξ is depicted on the Fig. 2.

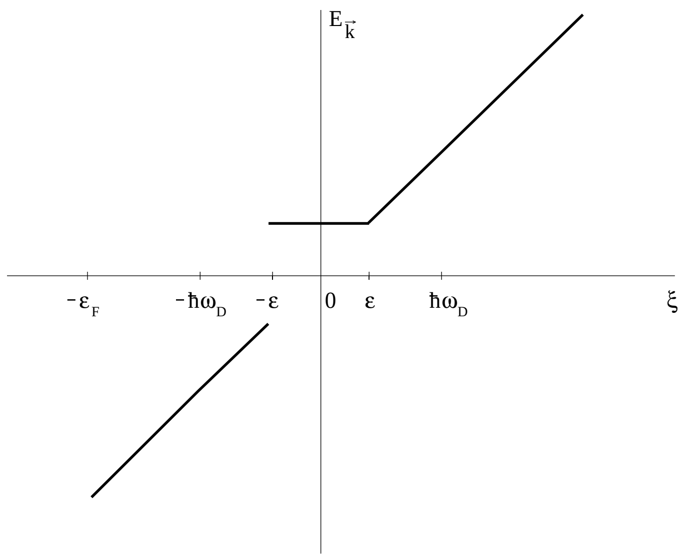

FIGURE 2. The energy spectrum $E_{\vec{k}}$ of quasiparticles as a function of ξ.

The energy spectrum $E_{\vec{k}}$ as given by the formulae (5.15) differs quantitatively from that one given by the relation (4.21) and depicted on the Fig. 1 corresponding to the standard solution of the BCS theory. The energy spectrum (5.15) has no discontinuities at fixed points, at $\mid \xi \mid = \hbar\omega_D$, in contrast to the standard solution of the BCS theory. It has a single discontinuity, at the point $\xi = -\epsilon$, which is a function of the temperature T.

The function $\epsilon(T)$ determined by the transcendental equation

$$1 = G\frac{2\epsilon_F}{\epsilon} \tanh \frac{\beta\epsilon}{2} \tag{5.16}$$

has the following behavior for the limiting cases

$$\epsilon(T) = \epsilon_0 \big(1 - 2e^{-\beta\epsilon_0}\big) \ , \ T \ll T_c \tag{5.17}$$

and

$$\epsilon(T) = \sqrt{3}\, \epsilon_0 \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}} \ , \ T_c - T \ll T_c \ , \tag{5.18}$$

where

$$\epsilon_0 \equiv \epsilon(T = 0) = 2G\epsilon_F = 2k_B T_c \ . \tag{5.19}$$

Thus the ratio $\frac{\epsilon_0}{k_B T_c} = 2$ is a universal constant, independent on material parameters of the thin layer under consideration.

For the numerical analysis of the solution $\epsilon(T)$ to the equation (5.16) it is convenient to introduce the following dimensionless variables

$$\eta(T) = \frac{\epsilon(T)}{\epsilon_0} \ , \ \tau = \frac{T}{T_c} \ , \tag{5.20}$$

where τ is the so called reduced temperature. In this notation the function $\eta(T)$ is expressed in the implicit form

$$\eta = \tanh\left(\frac{\eta}{\tau}\right) \tag{5.21}$$

and its numerical representation is shown on the Fig. 3.

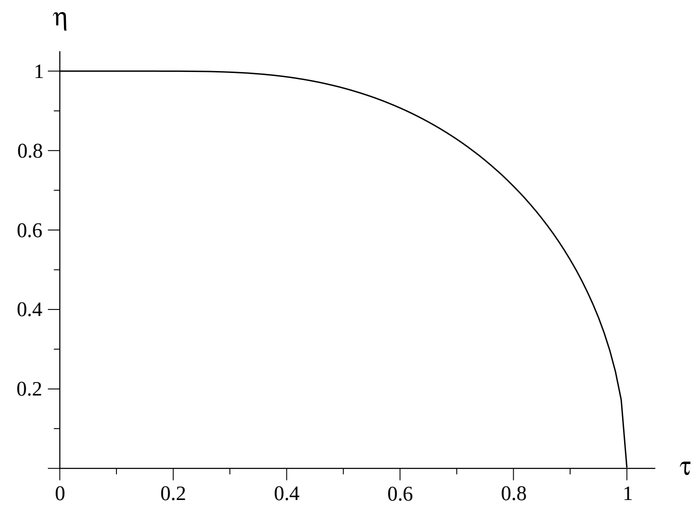

FIGURE 3. The energy spectrum $\epsilon(T) = \eta(\tau)\epsilon_0$ as a function of the reduced temperature τ.

From the relations (4.33) and (5.15) we get the explicit expression for the following average value

$$\left\langle a_{-\vec{k},-} a_{\vec{k},+} \right\rangle_{03} = \frac{1}{4G\epsilon_F}(\epsilon^2 - \xi_{\vec{k}}^2)^{\frac{1}{2}}\Theta(\epsilon- \mid \xi_{\vec{k}} \mid) \ \Theta\left(G - \frac{1}{\beta\epsilon_F}\right) \ , \ \epsilon < \hbar\omega_D \ , \tag{5.22}$$

which is of great importance for the calculation of thermodynamic properties of the novel superconducting state.

6. Thermodynamical properties of the superconducting state

All thermodynamic properties of the superconducting state are extracted from the difference (4.5) between the grand canonical potential $\Omega(T,V,\mu;g) \equiv \Omega_s$ corresponding to the superconducting state for $T < T_c$ and $\Omega(T,V,\mu;g=0) \equiv \Omega_n$ corresponding to the normal state. For this reason we first evaluate the average value $\langle H_I \rangle(g)$. By using the result (5.22) we get the expression

$$\langle H_I \rangle(g) = -\frac{g}{V}\,\frac{\Theta\big(G-\frac{1}{\beta\epsilon_F}\big)}{16G^2\epsilon_F^2}\sum_{\vec{k},\vec{k}'}(\epsilon^2-\xi_{\vec{k}}^2)^{\frac{1}{2}}(\epsilon^2-\xi_{\vec{k}'}^2)^{\frac{1}{2}}\ \Theta(\epsilon-\mid\xi_{\vec{k}}\mid)\ \Theta(\epsilon-\mid\xi_{\vec{k}'}\mid)+R(g)\ . \tag{6.1}$$

The above sum over $\vec{k}$ and $\vec{k}'$ is evaluated as an integral by the standard substitution

$$\sum_{\vec{k},\vec{k}'}\cdots \to \frac{V^2}{(2\pi)^6}\int d^3\vec{k}d^3\vec{k}'$$

and by using the constraints (5.10) we get the result

$$\langle H_I \rangle(g) = -\frac{1}{(2\pi)^2}\frac{gV}{2^5G^2\epsilon_F}\left(\frac{m}{\hbar^2}\right)^3\epsilon^4(T)\Theta\left(G-\frac{1}{\beta\epsilon_F}\right)+R(g)\ , \tag{6.2}$$

where $R(g)$ denotes all remaining terms in the perturbation series which are negligible in the thermodynamic limit $V\to\infty$ similarly as for the perturbation series with unperturbed Hamiltonians H_{01} and H_{02}.

Now, we are ready to express the difference (4.5) as the following integral

$$\Omega_s-\Omega_n = -\frac{Vg}{2^5k_BT_c}\frac{1}{(2\pi)^2}\left(\frac{m}{\hbar^2}\right)^3\int_{\frac{1}{\beta\epsilon_F}}^{G}\frac{dG'}{G'^2}\ \epsilon^4\ , \tag{6.3}$$

where G is the dimensionless coupling constant defined by equation (5.12). The last integral is luckily in a particularly convenient form because the relation (5.16) expresses G as the following function of ϵ,

$$\frac{1}{G} = \frac{2\epsilon_F}{\epsilon}\tanh\frac{\beta\epsilon}{2} \tag{6.4}$$

The direct substitution of the last relation into the integral (6.3) leads to the formula

$$\Omega_s-\Omega_n = \frac{V}{(2\pi^2)}\frac{g\epsilon_F}{16k_BT_c}\left(\frac{m}{\hbar^2}\right)^3\int_0^{\epsilon}d\epsilon'\epsilon'^4\frac{d}{d\epsilon'}\left(\frac{1}{\epsilon'}\tanh\frac{\beta\epsilon'}{2}\right)\ . \tag{6.5}$$

Two per partes integrations of the last integral and the use of the reduced variables $\eta=\frac{\epsilon}{\epsilon_0}$ and $\tau=\frac{T}{T_c}$ give us the result

$$\Omega_s-\Omega_n = -\frac{V}{6(2\pi)^2}\ g\epsilon_F(k_BT_c)^2\left(\frac{m}{\hbar^2}\right)^3\left[\eta^4-\frac{1}{2}\tau^3\varphi(\eta)\right]\ , \tag{6.6}$$

where $\varphi(\eta)$ is a function of η defined by the integral

$$\varphi(\eta) = \int_0^{\eta}dx\left(\ln\frac{1+x}{1-x}\right)^3\ . \tag{6.7}$$

By passing from the grand canonical potential $\Omega(T,V,\mu)$ to the Helmholtz free energy $F(T,V,N)$ we get the difference

$$\Omega_s-\Omega_n = F_s-F_n = -\frac{V}{8\pi}H_c^2(T)\ , \tag{6.8}$$

where $H_c(T)$ is the thermodynamic critical magnetic field given by the formula

$$H_c(T) = H_c(0)\left[\eta^4-\frac{1}{2}\tau^3\varphi(\eta)\right]^{\frac{1}{2}}\ . \tag{6.9}$$

Here, $H_c(0)$ is the critical magnetic field at $T = 0$,

$$H_c(0) = \left(\frac{g\epsilon_F}{3\pi}\right)^{\frac{1}{2}} \left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}} = 2\pi^2 \left(\frac{g}{6\pi}\right)^{\frac{1}{2}} N(0) k_B T_c \, . \tag{6.10}$$

The temperature dependence of the ratio $R_H(\tau) = H_c(T)/H_c(0)$ is shown on the Fig. 4.

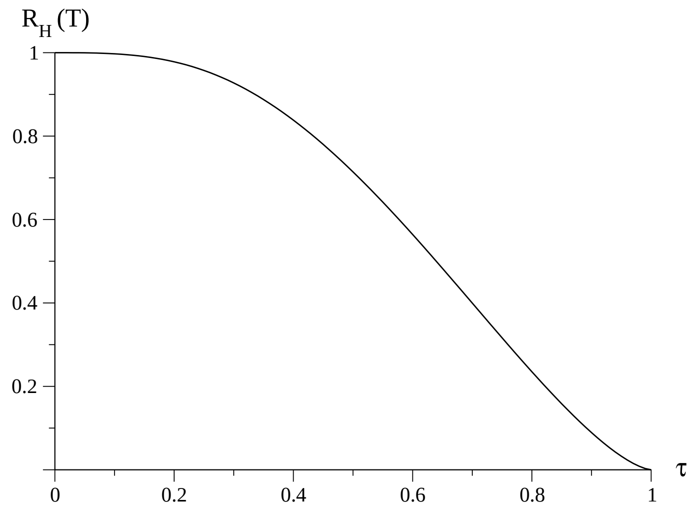

FIGURE 4. The temperature dependence of the critical magnetic field $H_c(T)$ plotted as the ratio $R_H(\tau) = H_c(T)/H_c(0)$.

The theoretical curve on the Fig. 4 is very similar to the experimental data of the upper critical magnetic field for HTS $Bi_2Sr_2CaCu_2O_8$ [10]. The behavior of $H_c(T)$ at $T_c - T \ll T_c$ and $T \ll T_c$ is given by the following formulae

$$H_c(T) = 3H_c(0)(1-\tau)^{\frac{3}{2}} \; , \; T_c - T \ll T_c \tag{6.11}$$

and

$$H_c(T) = H_c(0)\left(1 - \frac{9\zeta(3)}{4}\,\tau^3\right) \; , \; T \ll T_c \; , \tag{6.12}$$

where $\zeta(x)$ is the Riemann dzeta function, $\zeta(3) \approx 1.202$.

Now we compare the results (6.10)-(6.12) with the behavior of the critical magnetic field $H_c^{'}(T)$ corresponding to the results of the standard solution of the BCS theory [6]

$$H_c^{'}(0) = \pi e^{-\gamma} [4\pi N(0)]^{\frac{1}{2}} k_B T_c^{'} \doteq 1.76 [4\pi N(0)]^{\frac{1}{2}} k_B T_c^{'} \; , \tag{6.13}$$

$$H_c^{'}(T) = H_c^{'}(0)\, e^{\gamma} \left[\frac{8}{7\zeta(3)}\right]^{\frac{1}{2}} (1-\tau') \doteq 1.74 H_c^{'}(0)(1-\tau') \; , \; T_c^{'} - T \ll T_c^{'} \tag{6.14}$$

and

$$H_c^{'}(T) = H_c^{'}(0)\left[1 - \frac{e^{2\gamma}}{3}\,\tau'^2\right] \doteq H_c^{'}(0)\big[1 - 1.06\,\tau'^2\big] \; , \; T \ll T_c^{'} \, . \tag{6.15}$$

From the last formulae one sees qualitative differences between the critical magnetic fields $H_c(T)$ and $H_c^{'}(T)$ corresponding to the unconventional solution and to the standard solution resp. of the BCS theory. The curve $H_c(T)$ is convex for $\tau \in \left(0, \frac{1}{2}\right)$ and concave for $\tau \in \left(\frac{1}{2}, 1\right)$ while $H_c^{'}(T)$ of the standard solution is a convex curve for the full temperature interval $\tau' \in (0, 1)$.

We now calculate the specific heat anomaly from the difference (6.6) to get the result

$$\frac{C_s(T) - C_n(T)}{C_n(T)} = \frac{3}{8} g N(0) \left\{ \frac{2}{\tau^2} \frac{\eta^4(1-\eta^2)}{\eta^2 + \tau - 1} - \frac{1}{2} \tau \varphi(\eta) \right\} , \tag{6.16}$$

where

$$C_n(T) = V \frac{2\pi^2}{3} N(0) k_B^2 T \tag{6.17}$$

is the specific heat of an ideal electron gas. The temperature behavior of the specific heat anomaly (6.16) plotted as the ratio

$$R_C(\tau) = \frac{8}{3gN(0)} \frac{C_s(T) - C_n(T)}{C_n(T)} \tag{6.18}$$

is shown on the Fig. 5.

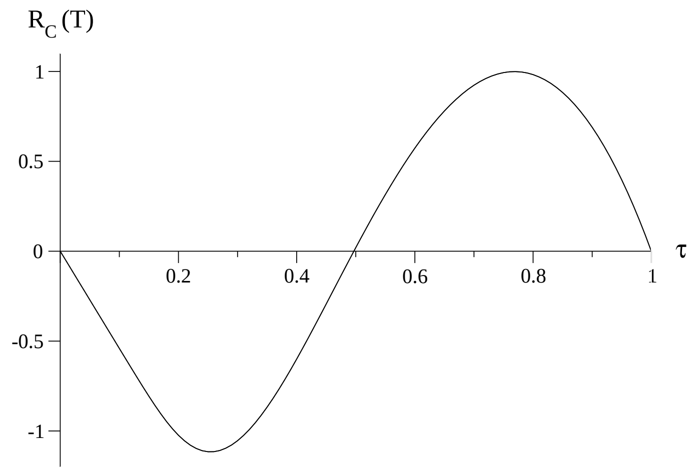

FIGURE 5. The specific heat anomaly corresponding to the unconventional solution of the BCS theory.

The specific heat anomaly (6.16) has the following behavior in the limits

$$\frac{C_s(T) - C_n(T)}{C_n(T)} = \frac{27}{8} g N(0)(1-\tau) \ , \ T_c - T \ll T_c \tag{6.19}$$

and

$$\frac{C_s(T) - C_n(T)}{C_n(T)} = -\frac{27\zeta(3)}{16} g N(0) \tau \ , \ T \ll T_c \ . \tag{6.20}$$

This behavior of the specific heat anomaly is qualitatively completely different from that corresponding to the standard solution [6] of the BCS theory

$$\frac{C_s^{'}(T) - C_n(T)}{C_n(T)} = \frac{12}{7\zeta(3)} + O\big[(1-\tau)^2\big] \doteq 1.43 + O\big[(1-\tau)^2\big] \ , \ T_c - T \ll T_c \tag{6.21}$$

and

$$\frac{C_s^{'}(T) - C_n(T)}{C_n(T)} = -1 + O\big(e^{-\frac{1}{\tau}}\big) \ , \ T \ll T_c \ . \tag{6.22}$$

The specific heat $C_s(T)$ corresponding to the unconventional superconductivity (6.16) is a continuous function of T at $T = T_c$, as is seen from (6.19), however, with a discontinuous derivative at $T = T_c$ in a sharp contradistinction to the standard case. Thus the unconventional solution considered here gives rise to a superconducting phase transition of the third order. From the results

(6.8)-(6.20) it is evident that the properties of superconducting states associated with two different inequivalent representations of the canonical anticommutator ring of the electron field operators (2.15) are qualitatively completely different.

Despite of the fact that we did not have any ambitions to explain some experimental data, but only to study the theoretical consequences coming from different inequivalent representations of the canonical anticommutator ring (2.15) within the framework of the BCS theory, we nevertheless mention a resemblance of the theoretical results (6.16)-(6.20) to existing experimental data. The specific heat anomaly on the Fig. 5 for $\tau \in \left(\frac{1}{2}, 1\right)$ is similar to the specific heat anomalies of cuprate HTS [11]. The shape of the specific heat anomaly as T approaches to zero, as given by (6.20), tells us that the specific heat $C_s(T)$ in this novel superconducting state behaves as a polynomial of T, i.e., similarly as electronic components of ordinary metals in the normal state. This behavior is again, perhaps accidentally, consistent with experimental data of cuprates HTS [11, 12] and is completely different from the behavior of LTS described by the standard solution of the BCS theory. We also recall that the specific heat jump at T_c for the standard solution of the BCS theory (6.21) is given by a universal number which is fairly consistent with experimental data for LTS. However, the novel superconducting state discussed here relates the specific heat anomaly (6.16) to the normal specific heat $C_n(T)$ by a material parameter $gN(0)$. This feature seems to be again in a correspondence with experimental data of HTS [12]. For example, the specific heat anomaly of $\mathrm{YBa_2Cu_3O_x}$ which ranks highest among HTS, does not exceed 5% of the normal specific heat. The situation for $\mathrm{Bi_2Sr_2CaCu_2O_x}$ is about three times worse. Thus it seems that the properties of the unconventional solution to the BCS Hamiltonian investigated in this paper may have some relevance for understanding of the superconductivity mechanism in HTS.

The system of electrons under consideration, however, stabilizes in a phase corresponding to the minimal Helmholtz free energy. Thus, for completeness, we analyze the stability of the novel superconducting phase resulting from the unperturbed Hamiltonian H_{03} with respect to the standard BCS phase arising from the unperturbed Hamiltonian H_{02}. The Helmholtz free energy in each of the superconducting phases is determined by the critical magnetic fields $H_c(T)$ and $H_c^{'}(T)$ as given by the relation (6.8). The critical magnetic field $H_c(T)$ for the novel superconducting phase is given by the relations (6.9)-(6.10). The temperature behavior of the critical magnetic field $H_c^{'}(T)$ corresponding to the standard superconducting phase can be approximated with a high accuracy by the formula

$$H_c^{'}(T) = H_c^{'}(0)(1 - \tau'^2) , \tag{6.23}$$

where $\tau' = \frac{T}{T_c'}$ and $H_c^{'}(0)$ is given by (6.13). It is convenient to introduce the "reduced" Helmholtz free energy difference Δf defined by

$$\Delta f \equiv \frac{8\pi(F_S - F_N)}{V H_c'^2(0)} \tag{6.24}$$

which has the following expression

$$\Delta f = -\frac{H_c^2(0)}{H_c'^2(0)} \left[\eta^4 - \frac{1}{2}\tau^3 \varphi(\eta) \right] \tag{6.25}$$

for the novel superconducting phase and

$$\Delta f^{'} = -(1 - \tau'^2)^2 \tag{6.26}$$

for the standard BCS result. The novel superconducting phase is preferable over the standard phase provided that the following inequality is satisfied:

$$\Delta f < \Delta f^{'} . \tag{6.27}$$

The last inequality is always satisfied if $\frac{H_c(0)}{H_c'(0)} > 0$, i.e., if the material parameters ω_D, ϵ_F, g and the thickness of the layers d satisfy the following condition

$$T_c > e^{-\gamma} \sqrt{\frac{6}{gN(0)}}\, T_c^{'} \,, \tag{6.28}$$

where T_c and $T_c^{'}$ should be expressed in terms of material parameters as given by (5.12)-(5.14) and (4.23) respectively.

Since we have the same Hamiltonian responsible for both the novel superconducting phase and for the standard BCS solution in two inequivalent representations of the anticommutator ring of electron field operators we may assume that the values of the coupling constant g and the density of electron states $N(0)$ at the Fermi surface are the same as those experimentally found for LTS [13]. Therefore, the values $gN(0) \in (0.1, 0.5)$ [13]. For the reasonably chosen values $T_c = 100$ K, $T_c^{'} = 20$ K and $gN(0) = 0.1$ the numerical results of temperature dependence of the reduced Helmholtz free energies Δf are shown on the Fig. 6.

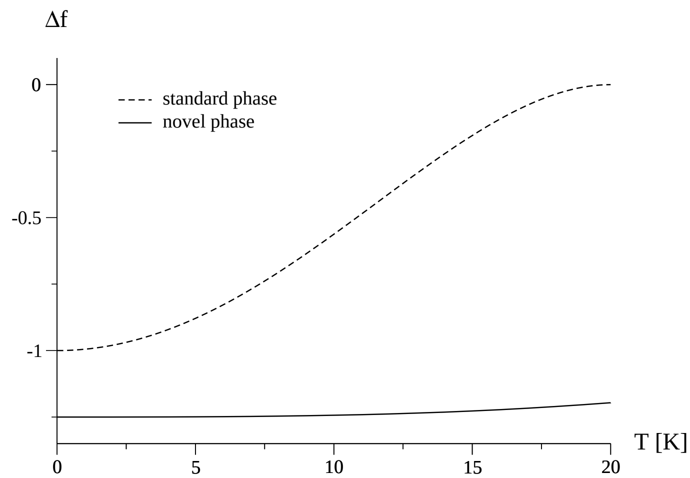

FIGURE 6. The comparison of Helmholtz free energies corresponding to the standard and the novel solution to the BCS theory for $T_c = 100$ K, $T_c^{'} = 20$ K and $gN(0) = 0.1$.

Fig. 6 clearly shows that the novel superconducting phase associated with the third inequivalent representation of the anticommutator ring of electron field operators of the same BCS Hamiltonian is energetically preferable with respect to its standard superconducting phase.

7. Summary

In applications of quantum field theory of many particle systems one has intuitively believed that a given Hamiltonian H determines uniquely a single Gibbs state of a system at given values of thermodynamic variables like temperature T, volume V and the number N of particles. Such a belief would be completely correct provided that the canonical commutation or anticommutation relations of field operators had a unique representation. In the thermodynamic limit $V \to \infty$, $N \to \infty$, $\frac{N}{V} \to$ constant, one deals, in fact, with systems having the infinite number of degrees of freedom. In this case, as it has been pointed out by Haag [1], the canonical commutation or anticommutation relations of field operators have no longer unique solutions, i.e., they have several different inequivalent representations. This fact has to be taken into account in all discussions concerning the quantum field theory of many particle systems. Namely, with each inequivalent representation of commutation or anticommutation relations one has to associate the corresponding "matrix form" of the Hamiltonian H leading to the corresponding grand canonical partition function Z and the grand canonical potential Ω which fully specify the Gibbs state of the system. This argument indicates the fact that the system can have as many Gibbs states as many inequivalent representations of the canonical commutator or anticommutator ring exist for the given Hamiltonian H.

To our best knowledge there is no systematic method for a classification of all inequivalent representations of the canonical commutator or anticommutator ring of field operators entering a given Hamiltonian H, e.g., as the method one has for the classification of the irreducible representations of Lie algebras. The only known way how one selects an inequivalent representation of the canonical commutator or anticommutator ring of field operators entering a given Hamiltonian H is to select a suitable unperturbed Hamiltonian H_0 which can be exactly diagonalized.

In the present work we have explicitly constructed three inequivalent representations of the canonical anticommutator ring of electron field operators entering the Hamiltonian of the BCS theory of superconductivity. Each inequivalent representation has been constructed by selecting the unperturbed Hamiltonian H_{01}, H_{02} or H_{03}. Each inequivalent representation specifies the corresponding Gibbs state of the system. The physical system, however, stabilizes in a Gibbs state having the minimal Helmholtz free energy at given values of the thermodynamic variables like temperature T, volume V and the particle number N. From this reason it is important to search for all relevant inequivalent representations of the canonical anticommutator ring of field operators entering the Hamiltonian of the system. The normal state and the standard superconducting state of the BCS theory corresponding to two inequivalent representations have been generally known long time ago. We have constructed the third inequivalent representation which has not been known till now. This representation can, perhaps, be relevant for describing the superconducting properties of HTS.

8. Acknowledgements

The authors are very grateful for many elucidating discussions with Professors C. Cronström, I. Hubač and R. Hlubina.

References

1. R. Haag, On quantum field theories, *Dan. Mat. Fys. Medd.* **29**, no 12 (1955).
2. J. Bardeen, L. N. Cooper and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
3. N. N. Bogoliubov, *Physics* **26**, S1 (1960).
4. P. Kalinay, M. Noga, *Czech. J. Phys.* **48**, 1615 (1998).
5. N. N. Bogoliubov, *Sov. Phys. - JETF* **41**, (1958);
   J. G. Valatin, *Nuovo Cimento* **7**, 843 (1958).
6. See e.g. A. L. Fetter and J. D. Walecka, Quantum theory of many-particle systems, McGraw-Hill Inc. New York, 1971, p. 396.
7. C. Cronström and M. Noga, *Czech. J. Phys.* **51**, 175 (2001).
8. Y. Uemura et al, *Phys. Rev. Lett.* **62**, 2317 (1989); **66**, 2665 (1991).
9. F. Pistolesi and G. C. Strinati, *Phys. Rev. B* **49**, 6356 (1994).
10. A. S. Alexandrov et al, *Phys. Rev. Lett.* **76**, 983 (1996).
11. A. Junod et al, *Physica B* **149-196**, 1497 (1994);
    A. Mirmelstein et al, *Physica C* **248**, 335 (1995);
    C. Meingast et al, *Physica C* **272**, 106 (1996).
12. A. Junod et al, *Physica C* **229**, 209 (1994).
13. See e.g. Superconductivity, Marcel Dekker Inc. New York, 1969, vol. 1, p. 122.